Abstract — This paper proposes a novel joint source coding and modulation scheme. The modulation constellation points are selected according to their prior symbol probabilities for better bandwidth as well as better bit error rate performance. Both the analysis and simulation results are presented to verify that the proposed scheme can achieve better performance than the conventional disjoint source coding and modulation schemes if the modulation and source coding are designed jointly and efficiently.

Index Terms—Quadrature Phase Shift Keying, Bit Error Rate, Maximum A Posteriori Probability, Maximum Likelihood.

I. INTRODUCTION

Typical inputs to a source encoder have unequal prior probabilities. For example, some alphabets have higher prior probabilities than other alphabets in an alphabet stream. However, the output of the conventional source encoder has been regarded as a bit stream with equal prior probability and fed into a channel encoder. Refer to Figure 1 (a). Then, a modulation scheme has been selected independently, regardless of the source unequal prior probabilities. For example, quadrature phase shift keying (QPSK) modulation takes two bits from the channel encoder output bit stream and maps them into a modulation symbol regardless of source alphabet prior probabilities. Hence, all the symbols or constellation points occur equally. However, the output symbols from the source encoder, e.g., Huffman source encoder, have unequal probabilities typically. Thus, the conventional modulation, e.g., QPSK, would not achieve optimum efficiency for the symbol stream with unequal probability. This is the motivation for this paper, which considers a joint source coding and modulation as shown in Figure 1 (b).

This paper proposes a novel joint source coding and modulation scheme that exploits the unequal symbol probability and introduces a novel distance criterion that is different from the one in [1], [2]. This paper also shows that the proposed joint source coding and modulation scheme with the maximum a posteriori probabilities (MAP) rule can achieve better performance than the conventional schemes, i.e., the disjoint source coding and modulation. We will compare performance between the conventional QPSK and the proposed scheme, called Y-Shape modulation, with the new distance criteria.

There have been many papers on joint source and channel coding, but very few on joint source coding and modulation, e.g., [1]. However, there are many differences between [1] and the proposed scheme in this paper. One of the main differences is that the one in [1] cannot be extendable for M-ary PSK with $M \geq 4$, whereas the proposed scheme can be applicable for any $M$-ary modulation.

On another note, in typical communication systems, a source encoder is followed by a channel encoder and then by modulation. However, the source encoder is followed by modulation in the proposed scheme. Hence, to use a channel coding, a non-binary symbol channel encoder, e.g., a $q$-ary low-density parity check encoder, can be employed after the joint source coding and modulation. However, the analysis requires M-dim integrals, which is not simple. In this paper, that difficulty will be solved. In the future, performance of the joint source coding and modulation including channel coding will be presented. This paper focuses on the joint source coding and modulation only.

II. SYSTEM MODEL

A. Y-3 Modulation in Two Dimensions

Y-3 modulation contains three symbols. An alphabet stream with unequal probability can be mapped into a binary bit stream by using the Huffman source coding algorithm. Table 1 shows an example of the source encoder, which maps a source alphabet stream of three alphabets $\{a,b,c\}$ into a bit stream of output bits $\{0,10,11\}$ or a constellation point stream of constellations $\{S_0,S_1,S_2\}$, with probabilities $\{0.5,0.25,0.25\}$, respectively.

The entropy of the output symbol from the source encoder can be computed as

$$H(X) = -\frac{1}{2} \log_2 1 - \left(\frac{1}{4} \log_2 \frac{1}{4}\right) \times 2 = 1.5 \text{ bits.}$$  

(1)

And the bandwidth efficiency of the joint source coding and modulation is
\[ \eta = \frac{1}{2} \times 1 + \frac{1}{4} \times 2 + \frac{1}{4} \times 2 = 1.5 \text{ bits/s Hz.} \quad (2) \]

Hence, the proposed scheme achieves the best bandwidth efficiency. And the signal constellation can be written as

\[ S_m = \left( \sqrt{E_s} \cos \left( \frac{2\pi}{3} m \right), \sqrt{E_s} \sin \left( \frac{2\pi}{3} m \right) \right) \quad (3) \]

where \( m = 0, 1, 2 \).

\[ y_1(x) = \sqrt{3}x + \frac{2\sigma^2}{\sqrt{3}E_s} \times \ln 2 \]

\[ y_2(x) = -\sqrt{3}x - \frac{2\sigma^2}{\sqrt{3}E_s} \times \ln 2 \]

\[ y_3(x) = 0. \]

\[ \left( \sqrt{E_s}, \sqrt{3E_s} \right) \]

\[ \left( \frac{E_s}{2}, \frac{\sqrt{3E_s}}{2} \right) \]

\[ S_1, S_2, S_0 \]

\[ S_1, S_2, S_3 \]

\[ \cdot \cdot \cdot \]

\[ Y_1, Y_2, Y_3 \]

\[ S_0, S_1, S_2 \]

Figure 2. Proposed signal constellations of Y-3 for the proposed joint source coding and modulation scheme.

Refer to Figure 3 for the optimum decision boundary equations.

Note that the crossing point of three optimum decision boundary lines is far to the left of the origin because three constellation points have unequal probability. This can be the cause for the improved performance, compared to the conventional scheme because the crossing point of the conventional disjoint source coding and modulation scheme is equal to the origin, and there is no room for improvement.

B. Y-shape and Z-shape in N-dimension

In [4], a look-up table was used for a high-dimensional case of up to 32 dimensions. However, this paper proposes a formula-based method for any \( n \)-dimensional signal constellation design.

In the \( n \)-dimension case, we could form a \( (n+1) \times n \)
matrix using the \( i \)-th signal constellation \( S_i \) in order to
determine the coordinates of each symbol where
\[
S_i = \{ S_{i1}, S_{i2}, S_{i3}, \ldots, S_{in} \}, \quad i = 1, 2, \ldots, n + 1. \tag{6}
\]

If the conventional Gram Schmidt procedure is used to
find the \( n \)-dimensional signal constellations, it takes too
much calculation and time in determining the locations of
the constellation points for large \( n \), e.g., \( n \geq 4 \). So, this paper
proposes a new technique for designing \( n \)-dimensional
signal constellations satisfying
\[
\sum_{j=1}^{n+1} S_{ij} = 0 \quad \text{and} \quad \sum_{j=1}^{n} S_{ij}^2 = E_s = 1 \tag{8}
\]
where the symbol energy was normalized to 1 in (8). So the
overall \( (n+1) \times n \) constellation matrix can be written as
\[
\begin{bmatrix}
S_{11} & 0 & 0 & \cdots & 0 \\
\frac{S_{12}}{\sqrt{1-S_{21}^2}} & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\frac{S_{1n}}{\sqrt{1-S_{2n}^2}} & \frac{S_{2n}}{n-1} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\frac{S_{1n}}{n} & \frac{S_{2n}}{n-1} & \cdots & \frac{S_{nn}}{n-2} & -S_{nn}
\end{bmatrix} \tag{9}
\]

Similar to the \( Y-3 \) proposed for two dimensions, the
decision boundary line between two symbols \( S_i \) and \( S_j \)
should satisfy
\[
(S_i x_1 - S_j x_1)_x + \cdots + (S_i x_n - S_j x_n)_x + \sigma^2 \ln \frac{P(S_i)}{P(S_j)} = 0.
\]
\[
(10)
\]
In four dimensions, we have two different branches,
which are \( Y-5 \) and \( Z-5 \). Their properties are listed in Table 2:

<table>
<thead>
<tr>
<th></th>
<th></th>
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<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>( S_0 )</td>
<td>0</td>
<td>0.5</td>
<td>0</td>
<td>0.5</td>
</tr>
<tr>
<td>( S_1 )</td>
<td>10</td>
<td>0.25</td>
<td>100</td>
<td>0.125</td>
</tr>
<tr>
<td>( S_2 )</td>
<td>110</td>
<td>0.125</td>
<td>10</td>
<td>0.125</td>
</tr>
<tr>
<td>( S_3 )</td>
<td>1110</td>
<td>0.0625</td>
<td>110</td>
<td>0.125</td>
</tr>
<tr>
<td>( S_4 )</td>
<td>1111</td>
<td>0.0625</td>
<td>111</td>
<td>0.125</td>
</tr>
</tbody>
</table>

III. PROBABILITY OF ERROR

Proakis derives the probability of the pair wise symbol
error in terms of the minimum distance between symbol \( i \)
and \( j \), and signal-to-noise ratio [5] as
\[
SER = P_y(E) = P(S_j | S_i) = kQ\left(\frac{d_{ij}}{\sqrt{2N_o}}\right) \tag{11}
\]
where \( Q(\alpha) \) is the tail probability, i.e., the integral of the
normal Gaussian density function from \( \alpha \) to the infinity.

So, the symbol error probability of the QPSK case is
\[
SER = 2Q\left(\sqrt{2SNR}\right) \tag{12}
\]
because the minimum distance of QPSK between two
constellations points is \( 2E_s = \sqrt{2E_b} \) and \( k = 2 \).

In the case of \( Y-3 \), first we determine the conditional
correct decision probability of each symbol [5], [6]. For
example, the correct decision probability for given symbol \( S_0 \) is
\[
P(C | S_0) = P \left( \begin{bmatrix} x, y \end{bmatrix} | x > \frac{2\sigma^2}{\sqrt{3}E} \text{ and } \frac{1}{2}\ln 2 \right)
\]
\[
\left[ -\frac{3x - 2\sigma^2}{\sqrt{3E}} \ln 2 < y \right] \tag{13}
\]
By applying normal distribution to the equation above, as
well as \( P(C | S_1) \) and \( P(C | S_2) \), we can obtain a general
equation as
\[
P(C | S_i) = \int_{-\infty}^{\infty} e^{-\frac{t^2}{2}} \times \left[ Q\left(-\sqrt{3}t + 3\ln 2\right) + a \times \frac{2\ln 2}{3\sqrt{3}SNR}\right]
\]
\[
\left[-Q\left(\sqrt{3}t - 3\ln 2\right) + b \times \frac{2\ln 2}{3\sqrt{3}SNR}\right] \right] dt \tag{14}
\]
where

<table>
<thead>
<tr>
<th>( t )</th>
<th>( a )</th>
<th>( b )</th>
<th>( c )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>-1</td>
<td>1</td>
</tr>
</tbody>
</table>

Using equation (14), we can find the average symbol error rate as
\[
P_e = 1 - \left( \frac{1}{2} P(C | S_0) + \frac{1}{4} P(C | S_1) + \frac{1}{4} P(C | S_2) \right). \tag{15}
\]

For the BER, we find the conditional probability of
symbol error that a symbol is mapped to another symbol as
\[
P(S_j | S_i) = \int_0^\infty e^{-\frac{t^2}{2}} \times \left(1 - Q\left(\frac{1}{\sqrt{3}}t - \frac{2\ln 2}{3\sqrt{3}SNR} + \sqrt{3}SNR\right)\right) dt \tag{16}
\]
In order to simplify the calculation for other symbols, since we know that the distance between two symbols is equal, we simply rotate other symbols to the position of $S_0$. Then,

$$P(S_o | S_1) = \frac{\ln 2}{3\sqrt{SNR}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} \times$$

$$\left(1 - Q\left(-\frac{1}{\sqrt{3}} t - \frac{2 \ln 2}{3\sqrt{3SNR}} - \sqrt{3SNR}\right)\right) dt$$

(17)

Then, the average BER can be written as

$$BER = \frac{\text{Total # of error bits}}{\text{Total # of transmitted bits}}$$

(22)

$$= \frac{1}{2} \left[ P(S_1 | S_0) \times 1 + P(S_2 | S_0) \times 1 \right]$$

$$+ \frac{1}{4} \left[ P(S_0 | S_1) \times 2 + P(S_2 | S_1) \times 1 \right]$$

$$+ \frac{1}{4} \left[ P(S_0 | S_2) \times 2 + P(S_1 | S_2) \times 1 \right]$$

$$= \frac{1}{4} \left[ P(S_0 | S_2) \times 2 + P(S_1 | S_2) \times 1 \right]$$

(23)

This can be compared to the BER of conventional QPSK as

$$BER = \frac{1}{2} \cdot SER$$

(24)

IV. ANALYSIS AND SIMULATION RESULTS

Figure 4 shows the theoretical and simulation results for the proposed joint source coding and modulation with Y-3 constellation. It is observed that the proposed Y-3 has lower BER than the conventional BPSK or QPSK, from 4 dB to high SNR, about 0.3 dB better than BPSK at BER equal to $10^{-5}$. It is also observed that the analysis results agree with simulation results. Figure 5 compares the BER results between the proposed Y-4 and the conventional BPSK or QPSK [7]. It is also observed that the proposed Y-4 has lower BER than the conventional BPSK or QPSK from 7 dB to high SNR, about 0.2 dB better than BPSK at BER equal to $10^{-5}$. Refer to Table 3.

**Table 3.** Comparisons of conventional disjoint source coding and QPSK/BPSK modulation, and the proposed joint source coding and Y-Shape modulation.

<table>
<thead>
<tr>
<th>No. of Symbols</th>
<th>QPSK</th>
<th>Y-4</th>
<th>Y-3</th>
<th>BPSK</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bandwidth Eff. $\eta$</td>
<td>2</td>
<td>1.75</td>
<td>1.5</td>
<td>1</td>
</tr>
<tr>
<td>Entropy</td>
<td>2</td>
<td>1.75</td>
<td>1.5</td>
<td>1</td>
</tr>
<tr>
<td>Minimum Dist. $d_{\text{min}}$</td>
<td>$\sqrt{(2Es)}$</td>
<td>$\sqrt{(2.67Es)}$</td>
<td>$\sqrt{(3Es)}$</td>
<td>$\sqrt{Es}$</td>
</tr>
<tr>
<td>Decision Rule</td>
<td>ML</td>
<td>MAP</td>
<td>MAP</td>
<td>ML</td>
</tr>
<tr>
<td>$E_s/N_0@BER=10^{-5}$</td>
<td>9.6 dB</td>
<td>9.4 dB</td>
<td>9.3 dB</td>
<td>9.6 dB</td>
</tr>
</tbody>
</table>

V. CONCLUSION

From Table 3, we can observe the advantages of the proposed joint source coding and modulation with Y-3 and Y-4 constellations over the conventional disjoint designs. The proposed scheme can be better than the conventional BPSK in both bandwidth efficiency as well as energy, i.e., BER. For higher modulation, e.g., $M$ larger than 4, we also observe that the proposed scheme can be better than QPSK in both senses. In the future, non-binary channel coding will be included in the proposed scheme, such as non-binary low-density parity check codes.

VI. ACKNOWLEDGEMENT

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REFERENCES

Figure 4. Simulation and theoretical BER/SER versus bit energy-to-thermal noise power spectral density ratio, $E_b/N_0$, in dB for the proposed joint source coding and modulation of Y-3 constellations.

Figure 5. Simulated BER/SER versus bit energy-to-thermal noise power spectral density ratio, $E_b/N_0$, in dB for the proposed joint source coding and modulation of Y-4 constellations.