Capacity and Cutoff Rate of Coded FH/MFSK Communications with Imperfect Side Information Generators

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Abstract—In a coded frequency-hopped M-ary frequency shift-keying (FH/MFSK) communication system, it is known that if perfect side information about the jamming state is available to the decoder, substantial improvement in performance is possible. However, thermal noise, which is present in practical communication systems, can corrupt the jamming state information (JSI). In this paper, when JSI is imperfect due to thermal noise, we calculate the capacities and cutoff rates of the channels as a function of the signal-to-jamming-noise ratio for memoryless, noncoherent FH/MFSK systems under partial-band noise jamming (PBNJ). We consider both soft and hard-decision metrics with perfect, imperfect, and no JSI. Also, we introduce three imperfect JSI generators. The first imperfect JSI generator uses the maximum a posteriori (MAP) decision rule based on the energy from an FH tone frequency which is near the M-signaling FH tone frequencies. The second decision rule utilizes the MAP rule, but it is based on the total energy received in the M-signaling FH tone frequencies. The third generator has the same decision statistics as the second generator, but its decision rule is an easily implementable suboptimum rule. If hard decisions are made and code rates are high, e.g., $\geq 0.7$, then the differences between the imperfect JSI generators and perfect JSI generator can be larger than 1 dB in the signal-to-jamming-noise ratio required to achieve the given capacity or cutoff rate, even though the thermal noise is quite small, e.g., a 25 dB signal-to-thermal noise ratio. If soft decisions are made, then the differences between the imperfect and perfect JSI cases are negligible.

I. INTRODUCTION

It is known that if perfect jamming state information (JSI) about the partial-band noise jamming (PBNJ) [to be defined in Section II] is available to the decoder of coded frequency-hopped M-ary frequency shift keying (FH/MFSK) systems, then substantial improvement in performance is possible, e.g., more than 5 dB improvement in signal-to-jamming-noise ratio required to achieve a 0.65 cutoff rate for a coded FH/binary FSK system with hard decisions and a 25 dB signal-to-thermal noise ratio [1]. However, thermal noise, which is present in practical communication systems, can corrupt the JSI. Thus, a JSI generator in practical environments cannot be perfect. For example, the probabilities of detection and false alarm are 0.81022 and 0.00963, respectively, for the JSI generator II (defined later) when the signal-to-thermal noise ratio is 25 dB and the cutoff rate of a coded FH/ BFSK system with hard decisions is 0.65 under PBNJ with a worst case jamming fraction. We need 2 dB more in the signal-to-jamming-noise required to achieve a 0.65 cutoff rate for this JSI generator, compared to that for the perfect JSI generator, even though the thermal noise is quite small.

In this paper, when JSI is imperfect due to the presence of thermal noise, we calculate the capacities and cutoff rates of the channels as a function of the signal-to-jamming-noise ratio for the memoryless, noncoherent FH/MFSK systems under PBNJ. We consider both soft- and hard-decision metrics with perfect, imperfect, and no JSI. Also, we introduce three imperfect JSI generators. The first imperfect JSI generator uses the maximum a posteriori (MAP) decision rule based on the energy from an FH tone frequency which is near to the $M$-signaling FH tone frequencies. The second decision rule utilizes the MAP rule, but it is based on the total energy received in the $M$-signaling FH tone frequencies. The third generator has the same decision statistics as the second generator, but its decision rule is an easily implementable suboptimum rule. If hard decisions are made, then the differences between the imperfect JSI cases are negligible.

II. CHANNEL MODEL

The block diagram of the communication system we consider is shown in Fig. 1. Existing papers regarding coding and spread spectrum communication systems, e.g., [6, vol. 1, ch. 4, and vol. II, ch. 2], [7]-[9], describe

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each block well. As shown in Fig. 1, the code symbols are transmitted over the coding channel outlined by dotted lines. Let $X$ denote an $M$-ary coding channel input symbol which takes values in the alphabet $\{1, 2, \cdots, M\}$. Assume that the interleaver and deinterleaver make the coding channel memoryless. The system uses the FH/MFSK signaling, and one MFSK symbol is transmitted per each hopping time interval. Let $B$ denote the bandwidth used by an FH tone frequency, which is the inverse of a hopping time interval. The frequency-hopped symbols are transmitted over a waveform channel which is affected by thermal noise and PBNJ. Assume that thermal noise is a white Gaussian process with two-sided power spectral density $N_0/2$ and that PBNJ is a bandlimited white Gaussian process. The receiver uses a synchronized version of the spreading sequence to dehop the received signals. The receiver contains noncoherent square-law envelope detectors. [Refer to Fig. 2(b).] In addition, let $Y = (Y_1, Y_2, \cdots, Y_M)$ denote a coding channel output observation vector, based on energy detections from $M$-ary signaling channels after frequency dehopping. Let the event $\{S=1\}$ mean that a transmitted code symbol is truly jammed, the event $\{S=0\}$ truly not jammed, $S$ an imperfect JSI from a JSI generator, $P_j$ the conditional probability that $\bar{S} = 1$ given $S = 1$ which is called the probability of detection, $P_{fa}$ the conditional probability that $\bar{S} = 1$ given $S = 0$ which is called the probability of false alarm, and $\rho$ the fraction of the full spread-spectrum bandwidth $W$ jammed by the PBNJ. Then the probability distribution of $S$ is $P\{S = 1\} = \rho$ and $P\{S = 0\} = 1 - \rho$. The probabilities of detection and false alarm depend on $\rho$, $N_0/2$ and the uniform noise jamming power spectral density $N_j/2 = J/2W$ where $J$ is the total jamming power received. We assume that $N_0$ and $N_j$ are constant parameters, and that the PBNJ minimizes the channel capacities or cutoff rates of coded FH/MFSK systems by varying $\rho$ for a given jamming power $J$. Then, the joint probabilities of $S$ and $\bar{S}$ are $P(S = 0, \bar{S} = 0) = (1 - \rho) (1 - P_{fa}(\rho))$, $P(S = 0, \bar{S} = 1) = (1 - \rho) P_{fa}(\rho)$, $P(S = 1, \bar{S} = 1) = \rho P_{d}(\rho)$, and $P(S = 1, \bar{S} = 0) = \rho (1 - P_{d}(\rho))$.

The conditional densities of $Y_i$, given $X$ and $\bar{S}$, are then

\[
p(y_i|X = j, \bar{S} = 1) = \alpha p(y_i|X = j, S = 1) + (1 - \alpha) p(y_i|X = j, S = 0)
\]

\[
p(y_i|X = j, \bar{S} = 0) = \beta p(y_i|X = j, S = 1) + (1 - \beta) p(y_i|X = j, S = 0),
\]

where

\[
\alpha = \frac{\rho P_d(\rho)}{(1 - \rho) P_{fa}(\rho) + \rho P_d(\rho)},
\]

\[
\beta = \frac{\rho (1 - P_d(\rho))}{(1 - \rho) (1 - P_{fa}(\rho) + \rho (1 - P_d(\rho))}.
\]

Note that if $P_d(\rho) = 1$ and $P_{fa}(\rho) = 0$ for any $\rho$ (i.e., perfect JSI is available), then $p(y_i|X = j, \bar{S} = k) = p(y_i|X = j, S = k)$. If $P_d(\rho) = 0.5$ and $P_{fa}(\rho) = 0.5$ for any $\rho$ (i.e., no side information is available), then $p(y_i|X = j, \bar{S} = k) = p(y_i|X = j)$ for $k = 0$ or 1. If $i$ is equal to $j$ (matched channels), then $p(y_i|X = j, S = k)$ is a noncentral chi-square density of two degrees of freedom with the noncentral parameter $2E, B/\sigma_i^2$ [12, eq. (4)-(7)], [1, eq. (17)] where $E, B$ is the received code symbol energy, $\sigma_i^2 = (N_0 + N_j/\rho)B$, and $\sigma_0^2 = N_0B$. If $i$ is not equal to $j$ (unmatched channels), then $p(y_i|X = j, S = k)$ is a central chi-square density of two degrees of freedom.

### III. IMPERFECT JSI GENERATORS

#### A. JSI Generator I

Assume that, among $M + 1$ FH tone frequencies, one FH tone frequency is dedicated to generating JSI, and $M$ FH tone frequencies are assigned to transmit an $M$-ary
coding channel symbol. Assume also that the dedicated FH tone frequency for JSI generation and the \( M \) signaling FH tone frequencies are hopped with the same frequency-hopping pattern. In addition, we can reasonably assume that if PBNJ jams any FH tone frequency in the \( M + 1 \) contiguous FH tone frequencies, then all \( M + 1 \) FH tone frequencies are jammed because an \( M + 1 \) contiguous frequency-hopping band is very small compared to \( W \). Other authors [1], [6]-[10] have made similar assumptions. The JSI generator I (as well as JSI generators II and III discussed later) is not extremely vulnerable to the PBNJ because the frequency-hopping pattern is not available to the jammer.

As shown in Fig. 2(a), after the frequency dehopper, JSI generator I measures energy received in the dedicated FH tone frequency. The measured energy is due only to thermal noise if the channel was not jammed or due to thermal noise plus jamming noise if the channel was jammed. If the measured energy, denoted by \( \xi \), is larger than or equal to the threshold value, then \( S = 1 \); otherwise, \( S = 0 \). We assume that all JSI generators in this paper have knowledge of the jamming fraction \( \rho \), and that JSI generator I uses the MAP rule to determine the threshold value. Using \( \sigma_0^2 = N_0 B \) and \( \sigma_0^2 = (N_0 + N_0/\rho)B \), the optimal threshold value is
\[
(4a)
\]
\[
\lambda_0 = \ln ((1 - \rho)/\rho).
\]
(Refer the MAP rule in [12].) The probabilities of detection and false alarm of JSI generator I are
\[
P_D(\rho) = \exp \left[ -\left( \lambda_0 + \ln \left( \sigma_0^2/\sigma_0^2 \right) \right)/\left( \sigma_0^2/\sigma_0^2 - 1 \right) \right]
\]
\[
P_{fa}(\rho) = \exp \left[ -\left( \lambda_0 + \ln \left( \sigma_0^2/\sigma_0^2 \right) \right)/(\sigma_0^2/\sigma_0^2 - 1) \right].
\]
For the discussion of numerical results in Section VI, we express our \( P_D(\rho) \) and \( P_{FA}(\rho) \) in terms of the symbol energy-to-jamming-noise ratio \( E_s/N_j \) and symbol energy-to-total noise ratio \( E_s/N_0 \) using \( \sigma_0^2/\sigma_j^2 = 1 + \{ E_s/N_0 \}/\{ E_s/N_j \} \). For our JSI generator I, the \( P_D(\rho) \) \( \geq P_{FA}(\rho) \) since \( \sigma_0^2 \geq \sigma_j^2 \). When there is no thermal noise \( \sigma_j^2 = 0 \), our JSI generator I becomes the perfect JSI generator with \( P_D(\rho) = 1 \) and \( P_{FA}(\rho) = 0 \) for any \( \rho \) between 0 and 1. Note that \( P_D(\rho) \) and \( P_{FA}(\rho) \) of JSI generator I do not depend on \( \rho \). In addition, note that the threshold of JSI generator I is easily implementable. It is, however, not bandwidth-efficient, especially for small \( M \).

For example, for \( M = 2 \), 1/3 of the total bandwidth is dedicated to obtaining side information. A more bandwidth-efficient method of obtaining JSI from energy measurements would be to base the decision on the total energy received in the \( M \) signaling FH tone frequencies instead of setting aside a dedicated tone frequency.

**B. JSI Generator II**

Fig. 2(b) illustrates JSI generator II, which is the same as the noncoherent FH/MFSK envelope-square demodulator, except for the energy summing device and the MAP decision rule device. Still, we assume a contiguous \( M \)-ary symbol per hop as in the previous JSI generator I. Let \( y \) denote the total energy (from \( M \) signaling FH tone frequencies) divided by the total power of the bandpassed thermal noise \( \sigma_0^2 = N_0B \). In addition, let \( p_1(y) \) denote the conditional probability density of \( y \) given the channel jammed (i.e., block of all \( M \) contiguous frequency tone slots is jammed), and \( p_0(y) \) the conditional probability density of \( y \) given the channel unjammed (i.e., block of all \( M \) contiguous frequency tone slots is unjammed). Then, for \( y \geq 0 \),

\[
p_1(y) = \frac{\sigma_0^2}{2\sigma_j^2} \left( \frac{y}{\sigma_j^2} \right)^{(M-1)/2} \exp \left( -\frac{y^2}{\sigma_j^2} - \frac{\sigma_0^2}{\sigma_j^2} I_{M-1} \left( \sqrt{\frac{\sigma_0^2}{\sigma_j^2} y} \right) \right)
\]

\[
p_0(y) = \frac{1}{2} \left( \frac{y}{\sigma_0^2} \right)^{(M-1)/2} \exp \left( -\frac{\sigma_0^2}{\sigma_j^2} - \frac{y}{\sigma_j^2} I_{M-1} \left( \sqrt{\frac{\sigma_0^2}{\sigma_j^2} y} \right) \right)
\]

where \( \sigma_j^2 = 2E_s/(N_0 + N_j) \), \( \sigma_0^2 = 2E_s/N_0 \), and \( I_{M-1}(x) \) is the modified Bessel function of order \( M - 1 \).

Let \( R(y) \) denote the optimum decision region based on the MAP rule for the jamming signal detection, which is \( R(y) = \{ y \} \) for \( p_1(y)/p_0(y) \geq (1 - \rho)/\rho \). If the total energy \( y \) belongs to \( R(y) \), then \( \hat{S} = 1 \); otherwise, \( \hat{S} = 0 \).

From the numerical analysis, we find that there are two types of decision regions, which depend on \( \rho, E_s/N_j, E_s/N_0 \), and \( M \). One is \( R(y) = \{ y \geq y_1^2 \} \) or \( y \geq y_2^2 \). The other is \( R(y) = \{ y \geq 0 \} \) \( y \geq y_2^2 \), which is a special case of the first type with \( y_1^2 = 0 \). The probabilities of detection and false alarm of JSI generator II are then

\[
P_D(\rho) = 1 - Q_M(a_1, b_1) + Q_M(a_1, b_2)
\]

\[
P_{FA}(\rho) = 1 - Q_M(a_0, b_0) + Q_M(a_0, b_2)
\]

where \( b_1 = \gamma_1^2 \sigma_0^2/\sigma_j^2, b_2 = \gamma_2^2 \sigma_0^2/\sigma_j^2, b_{i1} = y_1^2 \), \( b_{i2} = y_2^2 \), and \( Q_M(x, y) \) is the Marcum Q function. Note that the difference between the conditional average of \( y \) given the channel jammed and the conditional average of \( y \) given the channel unjammed is \( 2M(\sigma_1^2 - \sigma_j^2) \).

For our JSI generator I, the performance of JSI generator III becomes better as \( M \) increases. In addition, note that JSI generator II is bandwidth-efficient. It is, however, difficult to implement JSI generator II because we need the modified Bessel function of order \( M - 1 \).

**C. JSI Generator III**

Fig. 2(c) shows JSI generator III, which is the same as JSI generator II except for the decision device. JSI generator III uses a suboptimum decision rule: \( \hat{S} = 1 \) if the total energy \( y \) is larger than the threshold \( Th \); otherwise, \( \hat{S} = 0 \). The primary difference between JSI generator III and JSI generator II is that the threshold of generator III is chosen to be half the sum of the conditional average of \( y \) given the channel jammed, plus the conditional average of \( y \) given the channel unjammed, while the threshold of generator II is determined from the MAP rule. Then, the threshold of generator III \( Th \) is equal to two times average signal power plus \( M(\sigma_1^2 + \sigma_j^2) \). The probabilities of detection and false alarm of JSI generator III are then

\[
P_D(\rho) = Q_M(a_1, b_1)
\]

\[
P_{FA}(\rho) = Q_M(a_0, b_0)
\]

where \( b_1 = Th/\sigma_j^2 \) and \( b_0 = Th/\sigma_0^2 \). Note that, as we increase \( M \), the performance of JSI generator III also improves for the same reasons mentioned in the case of JSI generator II. In addition, note that JSI generator III is bandwidth-efficient and easily implementable if it has knowledge of the jamming fraction \( \rho \).

**IV. Channel Capacity**

In this section, we compute the channel capacity for memoryless, noncoherent FH/MFSK channels with PBNJ and thermal noise when JSI is imperfect. Since the coding channel is symmetric, the best probability distribution of \( X \) to maximize the mutual information between \( X \) and \( Y \), \( I(X; Y) \), is the uniform distribution on the alphabet [11, p. 64]. The channel capacity is

\[
C = \min_{0<\rho<1} \left\{ (\rho P_D(\rho) + (1 - \rho) P_{FA}(\rho)) I(X; Y) \bigg| \hat{S} = 1 \right\} + \left( \rho (1 - P_D(\rho)) + (1 - \rho)(1 - P_{FA}(\rho)) \right) \cdot I(X; Y) \bigg| \hat{S} = 0 \right\}.
\]
A. Capacity for Soft Decisions

If soft decisions are made at the demodulator, then the conditional mutual information $I(X;Y|\hat{S} = k)$ in (8) is

$$I(X;Y|\hat{S} = k) = \int p(y|X = x, \hat{S} = k) \log \frac{p(y|X = x, \hat{S} = k)}{p(y|\hat{S} = k)} dy.$$  

In the above integral, we can calculate the integrand, which is a function of the conditional probability densities, using McShane’s method. We do not include results on the capacity for soft decisions in this paper.

B. Capacity for Hard Decisions

If hard decisions are made at the demodulator, then the probability that the receiver makes an error on a symbol during a hop time interval is given as

$$p_I = \frac{1}{M} \sum_{j=2}^{M} (-1)^j \left( \begin{array}{c} M \\ j \end{array} \right) \exp \left\{ - \frac{E_j}{N_0 + N_j/p} \right\} \text{ for } S = 1$$

and

$$p_0 = \frac{1}{M} \sum_{j=2}^{M} (-1)^j \left( \begin{array}{c} M \\ j \end{array} \right) \exp \left\{ - \frac{E_j}{N_0} \right\} \text{ for } S = 0.$$

[1, eq. (19)]. The coding channel with hard decisions and perfect JSI is an $M$-ary symmetric channel with crossover probabilities $p_0$ and $p_1$. Also, the coding channel with hard decisions and imperfect JSI is an $M$-ary symmetric channel with crossover probabilities $\tilde{p}_1$ and $\tilde{p}_0$ where

$$\tilde{p}_1 = \alpha p_1 + (1 - \alpha)p_0$$

and

$$\tilde{p}_0 = \beta p_1 + (1 - \beta)p_0 \text{ if } \hat{S} = 0.$$  

We know that the conditional mutual information for the symmetric channel is given by

$$I(X;Y|\hat{S} = 1) = 1 + \left( 1 - \tilde{p}_1 \right) \log m(1 - \tilde{p}_1) + \tilde{p}_1 \log \left( \tilde{p}_1 / (M - 1) \right)$$

and

$$I(X;Y|\hat{S} = 0) = 1 + \left( 1 - \tilde{p}_0 \right) \log m(1 - \tilde{p}_0) + \tilde{p}_0 \log \left( \tilde{p}_0 / (M - 1) \right).$$

We can calculate the capacity for the channel with hard decisions and imperfect JSI by substitution of (10) into (8) in Section VI.

C. Inequalities Between Capacities

With imperfect JSI, the capacity is always less than or equal to the capacity with perfect JSI and always greater than or equal to the capacity without JSI, as summarized in the following theorems.

Theorem 1: Assume that soft decisions are made. Let $C_{\text{soft}}^\text{without}$ and $C_{\text{soft}}^\text{perfect}$ represent the channel capacity without and with perfect JSI, respectively, and let $C_{\text{soft}}^\text{imperfect}$ represent the channel capacity with any imperfect JSI. Then

$$C_{\text{soft}}^\text{without} \leq C_{\text{soft}}^\text{imperfect} \leq C_{\text{soft}}^\text{perfect}. \quad (11)$$

Proof: See the Appendix.

Theorem 2: Assume that hard decisions are made. Let $C_{\text{hard}}^\text{without}$ and $C_{\text{hard}}^\text{perfect}$ represent the channel capacity without and with perfect JSI, respectively, and let $C_{\text{hard}}^\text{imperfect}$ represent the channel capacity with any imperfect JSI. Then

$$C_{\text{hard}}^\text{without} \leq C_{\text{hard}}^\text{imperfect} \leq C_{\text{hard}}^\text{perfect}. \quad (12)$$

Proof: See the Appendix.

V. CUTOFF RATE

In this section, we compute the computational cutoff rate for memoryless, noncoherent channels with PBNC and thermal noise when JSI is imperfect. The cutoff rate is defined as

$$R_0 = 1 - \log m \frac{1 + (M - 1)D}{M} \text{ information symbols}$$

where parameter $D$ is a worst case (with respect to $p$) Chernoff bound on the probability that the metric value corresponding to the transmitted symbol $x$ is smaller than the metric value corresponding to a nontransmitted symbol $\hat{x}$ [6]-[9]:

$$D = \max_{0<p<1} \min_{0<y<1} \frac{1}{\lambda} \log \left( \frac{m(y; x; \hat{x})}{m(y; x; \hat{y})} \right) \text{ s.t. } y \neq x.$$  

In (14), "$E^{*}\"$ represents the expectation over the observed random vector $Y$ and $S$. The $m(y; x; \hat{x})$ is an additive metric, and $\lambda$ is the Chernoff bound parameter [6]. Given the channel output sequences, the decoder uses the metric to decide the sequence of the transmitted symbols.

The general relation of the parameter $D$ to the coded bit error probability is $P_b \leq G(D)$ where $G(\cdot)$ is a polynomial function determined solely by the specific code, whereas the parameter $D$ depends only on the coding channel and the decoder metric [6, vol. 1, pp. 194, 199].

A. Cutoff Rate for Soft Decision Metrics

When soft decisions are made, the metric we are considering is $m(y; x; \hat{x}) = c_\xi y_x$, which is a weighted version of the observed energy $y_x$ corresponding to input symbol $x$. In Section VI, we numerically optimize the weighting factors $c_\xi$ (which depend on the imperfect JSI) and $\lambda$. The parameter $D(\rho, \lambda)$ in (14) can be calculated...
as below:

\[ D(\rho, \lambda) = (1 - \rho) (1 - P_{FA}(\rho)) \frac{1}{1 - (\lambda c_0)^2} \]

\[
\cdot \exp \left[ - \frac{E_s}{N_0 (1 + \lambda c_0)} \right] \\
+ (1 - \rho) P_{FA}(\rho) \frac{1}{1 - (\lambda c_1)^2} \cdot \exp \left[ - \frac{E_i}{N_0 + N_j/\rho (1 + \lambda c_1)} \right] + \rho P_D(\rho) \frac{1}{1 - (\lambda c_1)^2} \cdot \exp \left[ - \frac{E_i}{N_0 + N_j/\rho (1 + \lambda c_1)} \right]
\]

\[ D(\rho) = 2 \sqrt{\{\rho A_1 + (1 - \rho)A_0\}\{\rho (1 - (M - 1)A_1) + (1 - \rho)(1 - (M - 1)A_0)\}} \\
+ \rho(M - 2)A_1 + (1 - \rho)(M - 2)A_0.
\]

\[ \beta_0 = (1 - \rho)(1 - P_{FA}(\rho))A_0 + \rho(1 - P_D(\rho))A_1, \]
\[ \beta_1 = (1 - \rho)P_{FA}(\rho)A_0 + \rho P_D(\rho)A_1 \]
\[ \gamma = (1 - \rho)(M - 2)A_0 + \rho(M - 1)A_1. \]

If perfect JSI is available, then \( D(\rho) \) in (16) becomes a known formula \([6, \text{vol. I, eq. (4.81)}]\), \( D(\rho) = \rho [2 \sqrt{A_1 (1 - (M - 1)A_1)} + (M - 2)A_1] + (1 - \rho) [2 \sqrt{A_0 (1 - (M - 1)A_0)} + (M - 2)A_0] \), and if no JSI is available.

From a convexity argument, we can state the following inequalities about cutoff rates for hard decisions:

\[ R_{\text{without hard dec.}} \leq R_{\text{imperfect JSI hard dec.}} \leq R_{\text{perfect JSI hard dec.}}. \]

VI. NUMERICAL RESULTS AND DISCUSSION

In this section, we present numerical results for the imperfect JSI generators discussed in Section III. We compare our results for these imperfect JSI cases to the results for both the perfect JSI case and no JSI case.

For channel capacity \( C \) in (8)-(10), we can determine the minimum value of the bit energy-to-jamming-noise ratio \( E_b/N_j \) for reliable communication as a function of the code rate \( r \) from the relation as \( E_b/N_j = C^{-1}(r)/(r \log M) \) where \( r \) is the code rate in \( M \)-ary units \([1]\). In Fig. 3, the \( E_b/N_j \) required to achieve the channel capacity is shown for BFSK with hard decisions when \( E_s/N_0 = 15 \text{ dB} \). The general behavior of the required \( E_b/N_j \) to achieve the channel capacity for other MFSK and \( E_s/N_0 \) is similar to that of the cutoff rate (to be shown later). Hence, only cutoff results will be presented below. If we use the cutoff rate as a measure of the channel performance, the analysis yields \( E_b/N_j \geq R_0^{-1}(r \log M) \) as the necessary condition for practical and reliable communications \([6]-[9]\).

A. Numerical Results for Soft-Decision Metrics Based on Cutoff Rate

In Fig. 4, the \( E_s/N_j \) required to achieve the cutoff rate is shown for binary and 8-ary FSK with soft decisions. For the case of soft decisions and no JSI, we know that the performance is poor \([6]\). However, notice that for soft decisions and imperfect JSI, results are very nearly as good as those for the perfect JSI case. This is explained as follows, and is a key result of this paper. Let \( \mu_a \) denote \( \lambda c_0 \) and let \( \nu \) denote \( \lambda c_1 \) in (15). To minimize \( D(\rho, \lambda) \) in (15) with respect to both \( \mu \) and \( \nu \), we can minimize the sum of the first term plus the third term with respect to \( \mu \), and we can separately minimize the sum of the second term plus the fourth term with respect to \( \nu \). Suppose that \( \mu_{\text{min}} \) and \( \nu_{\text{min}} \) minimize \( D(\rho, \lambda) \) for given \( P_{FA}(\rho), P_D(\rho) \), \( E_s/N_0 \), and \( E_s/N_j \). From the numerical results, we found that for any pair \( (P_{FA}(\rho), P_D(\rho)) \), the value of exp...
Without JSI
JSI GEN - I
JSI GEN - II
JSI GEN - III
PERFECT JSI

\[
\left( -\frac{E_i}{N_0} \mu (1 + \mu) \right) / (1 - \mu^2) \quad \text{at} \quad \mu = \mu_{\text{min}} \text{ is almost equal to the value of } \\
\left( -\frac{E_i}{N_0} \nu (1 + \nu) / (1 - \nu^2) \right) \quad \text{at} \quad \nu = \nu_{\text{min}} \text{, and the value of } \\
\left( -\frac{E_i}{(N_0 + N_j) / \rho} \mu (1 + \mu) / (1 - \mu^2) \right) \quad \text{at} \quad \mu = \mu_{\text{min}} \text{ is almost equal to the value of } \\
\left( -\frac{E_i}{(N_0 + N_j) / \rho} \nu (1 + \nu) / (1 - \nu^2) \right) \quad \text{at} \quad \nu = \nu_{\text{min}}. 
\]

Hence, the sum of the first and the second terms in (15) is almost independent of \( P_{\text{F}}(p) \), and the sum of the third and the fourth terms in (15) is almost independent of \( P_D(\rho) \). This implies that any imperfect JSI generator can achieve almost the same performance as the perfect JSI generator.

**B. Numerical Results for Hard-Decision Metrics Based on Cutoff Rate**

In Fig. 5, when hard decisions are made for BFSK, the \( E_b/N_j \) needed to achieve the cutoff rate are shown for two different values of \( E_i/N_0 \). The results for the 8-ary FSK case are shown in Fig. 6. Three main results are observed from Figs. 5 and 6.

First, we observe in Fig. 5 that as thermal noise power becomes weaker, the required \( E_b/N_j \) versus \( r \) for the imperfect JSI cases approaches that for perfect JSI cases. Second, Figs. 5 and 6 show that the difference between the perfect JSI generator and imperfect JSI generators can be significant. For example, in Fig. 5(b), even if thermal noise is very small (e.g., \( E_i/N_0 = 25 \text{ dB} \)), the imperfect JSI generator I is 1.11 dB worse than the perfect JSI generator, the JSI generator II is 2.04 dB worse, and the JSI generator III is 3.03 dB worse at a code rate of 0.7. Under the conditions in the above example, the optimum jamming fractions \( \rho^* \) are 0.3234 for JSI generator I, 0.2224 for JSI generator II, and 0.1464 for JSI generator III. The \( P_D \) and \( P_{\text{F}} \) at those optimum jamming fractions are 0.9408 and 0.5223 \times 10^{-2} \text{ for JSI generator I}, 0.8016 and 0.6849 \times 10^{-2} \text{ for JSI generator II}, and 0.5852 and 0.4241 \times 10^{-2} \text{ for JSI generator III}, respectively.

Third, in Fig. 5 for BFSK, we observe that JSI generator I is the best among the three JSI generators, JSI generator II is the second best, and JSI generator III is the worst in the sense of the bit energy-to-jamming-noise ratio required to achieve the cutoff rate. However, in Fig. 6, we observe that if \( M \) is larger, then JSI generators II and III can be better than JSI generator I (see Fig. 6(a) for a code rate larger than 0.6). This is because JSI generators II and III, which are based on the total energy in \( M \) signaling FH tones, improve as \( M \) increases, as we expect from (6) and (7), while generator I does not.

In Table I (\( M = 2 \text{ case} \)) and Table II (\( M = 8 \text{ case} \)), the \( E_b/N_j \) needed to achieve the cutoff rate is listed for eight different values of \( E_i/N_0 \) and for four different values of cutoff rate. An interesting observation in Table II is that JSI generator III (a suboptimum rule) can be better than JSI generator I (a MAP rule) in the \( E_b/N_j \) required to achieve the cutoff rate for BFSK with hard decisions when \( E_i/N_0 = 12.8 \text{ dB} \) (column 1) and 14.8 dB (column 2). We know that a detector based on a MAP rule is an

\[
\left( -\frac{E_i}{N_0} \mu (1 + \mu) \right) / (1 - \mu^2) \quad \text{at} \quad \mu = \mu_{\text{min}} \text{ is almost equal to the value of } \\
\left( -\frac{E_i}{N_0} \nu (1 + \nu) / (1 - \nu^2) \right) \quad \text{at} \quad \nu = \nu_{\text{min}} \text{, and the value of } \\
\left( -\frac{E_i}{(N_0 + N_j) / \rho} \mu (1 + \mu) / (1 - \mu^2) \right) \quad \text{at} \quad \mu = \mu_{\text{min}} \text{ is almost equal to the value of } \\
\left( -\frac{E_i}{(N_0 + N_j) / \rho} \nu (1 + \nu) / (1 - \nu^2) \right) \quad \text{at} \quad \nu = \nu_{\text{min}}. 
\]
Fig. 4. (a) $E_s/N_0$ needed to achieve cutoff rate for frequency-hopped binary FSK with soft-decision metrics when $E_s/N_0 = 15$ dB. (b) $E_s/N_0$ needed to achieve cutoff rate for frequency-hopped 8-ary FSK soft-decision metrics when $E_s/N_0 = 19.8$ dB.
Fig. 5. (a) $E_b/N_0$ needed to achieve cutoff rate for frequency-hopped binary FSK with hard-decision metrics when $E_s/N_0 = 15$ dB. (b) $E_b/N_0$ needed to achieve cutoff rate for frequency-hopped binary FSK with hard-decision metrics when $E_s/N_0 = 25$ dB.
Fig. 6. (a) $E_b/N_0$ needed to achieve cutoff rate for frequency-hopped 8-ary FSK with hard-decision metrics when $E_b/N_0 = 19.8$ dB. (b) $E_b/N_0$ needed to achieve cutoff rate for frequency-hopped 8-ary FSK with hard-decision metrics when $E_b/N_0 = 29.8$ dB.
TABLE I

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<tr>
<td>Without</td>
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TABLE II

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<th>ISI-III</th>
</tr>
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<tbody>
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<td>0.6</td>
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<tr>
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</table>

---

Proof of Theorem 1

The transition probabilities with imperfect JSI, \( p(y|x, S = 1) \) and \( p(y|x, S = 0) \), are convex combinations of the transition probabilities with perfect JSI, \( p(y|x, S = 1) \) and \( p(y|x, S = 0) \), as shown in (1) and (2). We can use a theorem [11, p. 29] which says the mutual information \( I(X; Y) \) is convex in the transition probabilities \( p(y|x) \). Thus, the mutual information between \( X \) and \( Y \) given imperfect JSI is

\[
I(X; Y | \hat{S} = 1) = \alpha I(X; Y | S = 1) + (1 - \alpha) I(X; Y | S = 0)
\]

and

\[
I(X; Y | \hat{S} = 0) = \beta I(X; Y | S = 1) + (1 - \beta) I(X; Y | S = 0).
\]

Applying the above two inequalities to (8) gives us

\[
C_{\text{imperfect}} = \min_{0 < \rho < 1} \{ \rho I(X; Y | S = 1) + (1 - \rho) I(X; Y | S = 0) \}.
\]

The right-hand side of the above inequality is just the channel capacity for soft decisions and perfect JSI, \( C_{\text{perfect}} \).

When soft decisions are made, the channel capacity without JSI is given by

\[
C_{\text{without}} = \min_{0 < \rho < 1} I(X; Y)
\]

\[
= \int p(y|x) \log \frac{p(y|x)}{p(y)} dy
\]

where

\[
p(y|x) = \rho p(y|x, S = 1) + (1 - \rho) p(y|x, S = 0)
\]

and

\[
p(y) = \rho p(y|S = 1) + (1 - \rho) p(y|S = 0).
\]

The imperfect JSI generator becomes a worst case generator when the probability of detection is 0.5 and the probability of false alarm is 0.5 for \( \rho \). Assume a worst case imperfect JSI generator. Then the conditional density of \( y \) given \( X \) and \( S \) becomes the conditional density of \( y \) given \( X \). This implies that the mutual information between \( X \) and \( Y \), given \( S \), becomes the mutual information between \( X \) and \( Y \). Hence, from (8), the channel capacity for a worst case imperfect JSI is

\[
C_{\text{imperfect}} = \min_{0 < \rho < 1} \{ I(X; Y | \hat{S} = 1) + \frac{1}{2} I(X; Y | \hat{S} = 0) \} = \min_{0 < \rho < 1} I(X; Y).
\]
Proof of Theorem 2

Assume the demodulator makes hard decisions. Let \( C(x) \) represent the capacity of an \( M \)-ary symmetric channel with crossover probability \( x \). Then

\[
C(x) = 1 + (1 - x) \log (1 - x) + x \log M - 1. \tag{A-6}
\]

The channel capacity for hard decisions and imperfect JSI is then from (10):

\[
C_{\text{hard imperfect}} = \min_{0 < \rho \leq 1} \left\{ \left( \rho P_D(\rho) + (1 - \rho) P_F A(\rho) \right) C(\bar{p}_1) + (1 - \rho) (1 - P_F A(\rho)) C(\bar{p}_0) \right\}. \tag{A-7}
\]

Since the capacity is convex \( U \) in the transition probabilities \( p_1 \) or \( p_0 \) [11, theorem, p. 29], the capacity is then

\[
C(\bar{p}_1) = C(\alpha p_1 + (1 - \alpha) p_0) \leq \alpha C(p_1) + (1 - \alpha) C(p_0) \tag{A-8}
\]

and

\[
C(\bar{p}_0) = C(\beta p_1 + (1 - \beta) p_0) \leq \beta C(p_1) + (1 - \beta) C(p_0). \tag{A-9}
\]

Applying the above two inequalities to (A-7) yields

\[
C_{\text{hard imperfect}} \leq \min_{0 < \rho \leq 1} \left\{ \rho C(p_1) + (1 - \rho) C(p_0) \right\} = C_{\text{hard perfect}}. \tag{A-10}
\]

When no JSI is available, an equivalent optimum jammer's strategy is to choose \( \rho \), maximizing the error probability instead of minimizing the mutual information since \( C(x) \) is a decreasing function of the error probability when the error probability is less than \((M - 1)/M\). The average of the error probabilities for \( S = 1 \) and \( S = 0 \) gives the error probability of the coding channel. Thus, the channel capacity without JSI is then

\[
C_{\text{hard without JSI}} = C \left( \max_{0 < \rho \leq 1} \left( \rho p_1 + (1 - \rho) p_0 \right) \right) \leq \min_{0 < \rho \leq 1} C(\rho p_1 + (1 - \rho) p_0). \tag{A-11}
\]

The above inequality follows because \( C(x) \) is a decreasing function. The right-hand side of the above inequality is just equal to the channel capacity for hard decisions and imperfect JSI when \( P_D(\rho) = 0.5 \) and \( P_F A(\rho) = 0.5 \). We know that a worst case imperfect JSI generator produces \( P_D(\rho) = 0.5 \) and \( P_F A(\rho) = 0.5 \) for any \( \rho \). This completes the proof of Theorem 2.

References


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