The qRAM Quantum Weightless Neuron Node
Elitist Dynamics

Fernando M. de Paula Neto¹, Wilson R. de Oliveira², Adenilton J. da Silva¹,², and Teresa B. Ludermir¹

¹ Centro de Informática, Universidade Federal de Pernambuco, Recife, PE, Brazil {fmpn2, tbl}@cin.ufpe.br
² Departamento de Estatística e Informática, Universidade Federal Rural de Pernambuco, Recife, PE, Brazil, wilson.rosa@gmail.com, ajs@deinfo.ufrpe.br

Abstract. We introduce two models for the quantum weightless neuron node (qRAM) dynamics. After proving the fact that it is not generally possible to mathematically isolate the second qubit after the application of a U-Controlled Operation over a couple of qubits, we propose dynamics models that operate on restrict set of states - the elitist states - during the iterations. The models are experimentally and analytically analysed in a dynamics with quadratic nonlinear operators. The proposed models exhibit sensitivity to initial conditions and chaotic behaviour.

1 Introduction

In the circuit model of computation (classical or quantum) iteration is implemented as an effective procedure generating circuits of appropriate size from a uniform family of circuits computing the task up to that size. Sometimes iteration is implemented by repeated applications of given circuit as e.g. Grover algorithm which is the Θ(√n) repetition of the Grover operator G [2] implemented as an acyclic quantum circuit.

Iterative behaviour is the essence of classical computing, and some physical systems are intrinsically cyclic, as the control system of a quantum robot [15] [16] interacting with the environment for navigation or identification, where a quantum computer controls its operations.

The dynamics of quantum cyclic networks can be studied via their operators, its phase analysis, for extracting eigenvalues and eigenstates in [18].

In the context of quantum information processing, Lloyd [17] showed that information in cyclic network benefits when there is no measurement. Nevertheless, the unitarity of the quantum transformations generally prevent the sensitivity to the initial conditions, although there are exceptions [9]. Measuring affects the dynamics of the quantum systems [8] as nonlinearity can emerge. This nonlinear behaviour has serious consequences in the dynamics bringing chaotic patterns into consideration.

³ See for instance Section 8.13 of the textbook [5].
In this work we show through a set of experiments and analysis that the qRAM quantum weightless neuron node demonstrates acute sensitivity to the initial conditions and chaotic behaviour. For that, we have used the quantum operator of the qRAM node and a nonlinear operator created by Bechmann et al. [7]. Although this operator is not a linear transformation (and does not preserve trace), it is physically realisable, notwithstanding our technological shortcoming to do so at the present moment. The qRAM node was introduced in [12] and [13] and has been further studied in [3,4] as a quantisation of logical neural node [11] and, to date, the only truly quantum neural networks model in the literature.

We prove that the target qubit after a controlled operator is not always decomposable as a product of two isolated quantum states, i.e. is entangled. So, two dynamic models in which only some states - elitist states - are used during the iterations is proposed and observe high sensitivity to initial conditions. After that, the results are analysed under the perspective of Julia Sets and Orbit Diagrams.

## 2 Dynamical Systems

It is usually necessary to understand the behaviour of systems as time evolves. This iterative process is the subject of the field Dynamical Systems where there are many tools and concepts that help designers and engineers to investigate the temporal behaviour of systems. Some of the concepts are presented here, in this section, to help the understanding and evaluation of the models that will be investigated in this work.

### Orbits

There are many problems in Science in general and in Mathematics in particular that involve iteration. Iteration means to repeat a process many times. In dynamics the process that is repeated is the application of a function. The result of the application of a function in previous time is used as input in the same function in the current time.

Given $x_0 \in \mathbb{R}$, we define the orbit of $x_0$ under $F$ to be the sequence of points $x_0, x_1 = F(x_0), x_2 = F^2(x_0), \ldots, x_n = F^n(x_0), \ldots$. The point $x_0$ is called "seed" of the orbit.

Sometimes it is useful to deal with a family of functions parametrized by a constant and so it is normal to represent it as $F_c(z)$ where $c$ is the constant. As example, we have $F_c(z) = z^2 + c$, and $F_2(z) = z^2 + 2$, where $c = 2$.

### Orbit Diagram

Images can be one of the most instructive tool, or the most intriguing one, to analyse dynamical systems. The orbit diagram allows us to capture the dynamics of a function $Q_c$ to different values of $c$, in the same figure. The resulting plot is
a valuable tool for the analysis of the dynamics of a family of functions as well it gives us the idea when $Q_c$ makes its transition to chaos.

In the orbit diagram, the parameter $c$ is plotted in the horizontal axis and the asymptotic orbit is plotted in the vertical axis. By asymptotic orbit it is understood the last values generated by the orbit, in other words, the values generated in the first iterations are discarded. Only the ones who overcome the transient of the function are kept. We can see the orbit diagram of the function $Q_c(z) = z^2 + c$ in the Figure 1.

![Orbit Diagram](image1)

Fig. 1: Orbit Diagram of the $Q_c(z) = z^2 + c$ function, where $-2 < c < 0.25$

### Julia Set

Julia Set is the place where the chaotic behaviour of a complex function occurs. [6]. For example, the squaring map $Q_0(z) = z^2$ is chaotic in the unit circle, because if $|z| < 1$ so $|Q_0^n(z)| \rightarrow 0$, when $n \rightarrow +\infty$, and if $|z| > 1$ so $|Q_0^n(z)| \rightarrow +\infty$ when $n \rightarrow +\infty$. $Q_0(z)$ is a notation of the orbit 0 to $Q(z)$.

Through the Filled Julia Set is possible to identify fractals whose nature is very quirk. The filled julia set of $Q_c$ is the set of points whose orbits are limited. The Julia Set of $Q_c$ is the limit of the filled Julia Set. $K_0 = \{z||z| \leq 1\}$ and $J_0 = \{z||z| = 1\}$ are, respectively, Filled Julia Set and Julia Set examples. An example of Filled Julia Set can be seen in Figure (2).

![Filled Julia Set](image2)

Fig. 2: Filled Julia Set of the $Q_c(z) = z^2 + c$, $c = -0.39054 - 0.58679i$

### Quantum Computing

A quantum bit is a unity two-dimensional vector in the computational basis $|0\rangle = [1,0]^T$ and $|1\rangle = [0,1]^T$. Any qubit $|\psi\rangle$ can be written as the linear combination
\[ |\psi\rangle = \alpha |0\rangle + \beta |1\rangle, \text{ where } \alpha \text{ and } \beta \text{ are complex numbers with } |\alpha|^2 + |\beta|^2 = 1. \]

Tensor product are used for composite systems \[ |ij\rangle = |i\rangle \otimes |j\rangle. \]

The tensor product of two spaces say \( A \) and \( B \), \( A \otimes B \), is the space having as basis tensor products of basis from \( A \) with basis from \( B \). Not always a state in \( |\psi\rangle \in A \otimes B \) can be decomposed as the tensor product of two states from \( A \) and \( B \). These indecomposable states are called entangled states. Examples of two qubits entangled states are the Bell states:

\[ |\Phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}, \quad |\Phi^-\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}}, \quad |\Psi^+\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}}, \quad |\Psi^-\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}} \]

Quantum operator \( U \) over \( n \) qubits is a unitary complex matrix of order \( 2^n \times 2^n \).

For example, some operators over 1 qubit are: \( I \) (Identity), \( X \) (NOT) and \( H \) (Hadamard), described below in Equation (1) and Equation (2) in matrix form and operator form. The combination of these unitary operators forms a quantum circuit.

\[
I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad I|0\rangle = |0\rangle \quad X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad X|0\rangle = |1\rangle
\]

\[
H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \quad H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)
\]

The CNOT operator has 2 inputs and 2 outputs and flips the second one if the first is 1, as showed in the Figure 3.

![Fig. 3: CNOT Operator](image)

The operators of quantum computation can be seen as special kinds of linear transformations, as matrices that operates in a vectorial basis. These special matrices are unitary and invertible [10].

### 4 Weightless Neural Network

The Weightless Neural Network (WNN), firstly proposed and investigated by Igor Aleksander [1], are the models of neural computation which have binary inputs and outputs and there are no weights between their nodes. They work as a lookup tables. Their supervised or non supervised learning is the updating of the contents of the lookup table entries trough the training set. This kind of learning is fast mainly because one can not modify the entries previously presented - in the weighted models, during the training, a given input-output can change the content learned before. It can be flexible, easy implemented using...
commercially available Random Access Memories (RAMs), parallelizable and has many applications due to be capable of generalization and pattern identification [11].

There are many models of weightless neuron nodes. We describe one of them in this work. A detailed analysis of the other models can be found in [11].

4.1 RAM Node

RAM Node was not conceived under biologic inspiration. Its working does not involve bio-behavioural, synapses, neuron signal electrical propagation. Its inspiration came from the easy of use of the RAM memories by the engineering to solve problems of pattern recognition.

The working of the RAM Node is given for the access to one entry in a lookup table of a RAM, in which the addressing is done by input bits. N input bits are capable to address $2^N$ positions of memory. Given a signal $x = x_1x_2x_3...x_N$, the output of the RAM Node is a bit $y = C[x]$, 0 or 1, stored in the memory position $x$. This behaviour can be viewed as a mapping of a logic function, where the node RAM compute any boolean function of its entries.

![RAM Node Diagram]

Fig. 4: RAM Node. Given an input $x$, the RAM Node outputs a bit $y$ stored in the $C[x]$.

The learning of the RAM Node is understood as the process to update the bit in the memory position of each one of the patterns in the learning set. The RAM Node can compute all binary function of its input while the weighted neuron node can compute only linearly separable functions.

5 Quantum Weightless Neuron Node

There are many models of quantum neurons in the literature that work with intricate quantum strategies and others with only quantum inspired strategies. The non-parallelism in these models or the use of non unitary operators requires that the software/hardware designer must carefully analyse these operation when they become implemented. De Paula Neto et.al. proposed an analysis of these models under this point of view [4].
In this section we present the qRAM Node, the quantization of the RAM Node from the previous section.

### 5.1 qRAM Node

de Oliveira et.al. in [12] and [13] define a quantization of the RAM neuron described in [1]. This model received a mathematical interpretation that, from it, one can generalize under the quantum point of view.

The RAM Node stores in one position of memory one unique bit. The initial description in the qRAM Node [13] shows that the matrix can represent the behavior of a classical RAM with one input and one output, without loss of generality:

\[
A = \begin{pmatrix}
I & 0 \\
0 & X
\end{pmatrix}
\quad \text{where}
\]

\[
A\langle 00 \rangle = |0\rangle I|0\rangle \\
A\langle 10 \rangle = |1\rangle X|0\rangle 
\]

This matrix returns 0 ou 1 depending of the first bit. The RAM of \(n\) input bits has a set of \(2^n\) of that \(A\)'s, consequently it has \(2^n\) selectors and one qubit of output \(|o\rangle = |0\rangle\). A representation of the circuit can be seen in Figure (5). The matrices \(A\)'s are implicit in the notation of the circuit by the CNOT operator, that chooses the operator \(U_0\) if the input \(|ab\rangle = |00\rangle\) or it chooses \(U_1\), if the input is \(|ab\rangle = |01\rangle\). This notation is described by the author as a q-ROM, since the operator \(U\) is fixed in the circuit and they can not be changed. In other words, the qubits of inputs \(|ab\rangle\) do the selection of which of the operators \(U\) will be chosen as the operation. These \(U\) are updated in the training step.

![Fig.5: q-ROM Node in the quantum circuit description, where \(U_i, i = 0, 1, ..., 2^n - 1\), can be the I or X Operators](image)

Training the ROM presented in Figure (5) is to choose the matrix \(U_i\) in each one of the \(i\) position. This RAM is extended to a representation in a quantum circuit that includes selectors, it is shown in Figure (6). In this way the training step changes the values of the selectors. da Silva et.al. propose a classic and quantum learning algorithm for the qRAM in [3].

For example, given the input \(|i\rangle\), the neuron will choose the selector \(|s_i\rangle\), that will be applied with the output qubit to the CNOT gate. If the input is in superposition of states, the neuron will choose the set of selectors in superposition
Fig. 6: Quantum circuit representing the qRAM Node to n=2 input qubits

and will apply this superposition in the output qubit. The qRAM neuron is represented by the operation:

$$\sum_{i=0}^{2^n-1} |i\rangle_n \langle i| A_{s_i}$$ (4)

6 Chaotic Models of the Dynamics

The unitarity of the quantum transformations generally prevents the exponential sensibility to the initial conditions, although there are exceptions [9]. However, the measurement in quantum systems affects its dynamics [8] and a non-linear behaviour can emerge from the system. Bechmann et.al., in [7], propose a non-linear quantum transformation and they argue that though this transformation is non-linear (and it does not preserve the trace), it is physically realizable.

Before we present our model, we explain the workings of the dynamics over one qubit proposed by Kiss et al. [8]. After that, we prove that it is not possible to verify completely the quantum state of the target bit after a U-Operator over 2 qubit. So the elitist method is presented and analysed both analytically and experimentally.

6.1 Model of Dynamics over one qubit

Kiss et al. [8] propose an analysis of the dynamics over one qubit of the non-linear operator proposed by Bechmann et al. [7]. This operator is employed to differentiate optimally between non-orthogonal spin-1/2 states. This dynamics is mapped by Equation (5), where $N$ is the function that normalizes the qubit with a factor. In this qubit, the factor is $1/\sum \rho_{ij}^2$.

$$\rho = S\rho, \quad \rho_{ij} \xrightarrow{S} N\rho_{ij}^2$$ (5)

So, if we have one qubit, the transformation $S$ proposed by Bechmann et al. is:
\[ |\psi\rangle_{\text{input}} = \alpha |0\rangle + \beta |1\rangle \xrightarrow{S} |\psi\rangle_{\text{output}} = N(\alpha^2 |0\rangle + \beta^2 |1\rangle) \] (6)

Kiss et al. [8] propose to include this transformation \( S \) during the dynamics over one qubit. The rotation operator \( U \) is a generic rotation operator in the Hilbert space, with \( x \) and \( \phi \) variables, as showed in the Equation (7).

\[
U(x, \phi) = \begin{pmatrix}
\cos(x) & \sin(x)e^{i\phi} \\
-sin(x)e^{-i\phi} & \cos(x)
\end{pmatrix}
\] (7)

We describe below in details the process of each step of the dynamics proposed by Kiss et al for the sake of complete understanding the proposed model. Given \( \psi \), an initial pure state:

\[
|\psi\rangle = N(|z\rangle|0\rangle + |1\rangle)
\] (8)

where the renormalization factor of \( N \) is \( \frac{1}{\sqrt{1+|z|^2}} \). The quadratic operator \( S \) [7] and a generic rotation operator are applied. The application of these operations are showed below:

\[
|\psi\rangle = N(|z\rangle|0\rangle + |1\rangle) \xrightarrow{S} |\psi_2\rangle = N(z^2|0\rangle + |1\rangle) \\
U|\psi_2\rangle = |\psi_3\rangle = N((z^2\cos(x) + \sin(x)e^{i\phi})|0\rangle + \\
+ (-\sin(x)e^{-i\phi}z^2 + \cos(x))|1\rangle)
\] (9)

This state \( |\psi_3\rangle \) is the quantum state after the first iteration. To recovery the \( z \) value after this dynamics, it is necessary to translate the state to the original format of pure state.

\[
|\psi_{\text{output}}\rangle = N(z'|0\rangle + |1\rangle)
\] (10)

If we get \( |\psi_3\rangle \) and divide the \( |0\rangle \) amplitude value for the \( |1\rangle \) amplitude value, we have the value of \( z \) after one iteration:

\[
z' = \frac{z^2\cos(x) + \sin(x)e^{i\phi}}{-\sin(x)e^{-i\phi}z^2 + \cos(x)}
\] (11)

It is easy to see that the normalization rate does not need to be considered because it will be cancelled after the division of the amplitudes.

Considering \( p = \tan(x)e^{i\phi} \), we have the analytic formula of this dynamical model that is studied in [8]:

\[
F_p(z) = \frac{z^2 + p}{-p^*z^2 + 1}
\] (12)
The qRAM Node is composed of a circuit with various CNOT’s. We will show here that we do not have a canonical way to always recover the target qubit after a CNOT Operator or after any U-Controlled Operator.

**Theorem 1.** Superposed target qubits that are controlled for the family of U-Controlled operators are entangled.

Proof: If we consider the qubits $|u\rangle = (a b)^T$ and $|v\rangle = (c d)^T$: If we apply to this state an U-Controlled operation, where $U$ is a generic rotation operator:

$$U(\alpha) = \begin{pmatrix} 
\cos(\alpha) - \sin(\alpha) \\
\sin(\alpha) \cos(\alpha) 
\end{pmatrix} \quad (13)$$

we get:

$$A_U|u\rangle|v\rangle = \begin{pmatrix} 
ac \\
ad \\
bc * \cos(\alpha) - bd * \sin(\alpha) \\
bc * \sin(\alpha) + bd * \cos(\alpha) 
\end{pmatrix} \quad (14)$$

The qubit $|u\rangle$ should not change for this operation and it seems simple to think that the second qubit $|v\rangle$ can be recovered in its quantum state. It is not possible in general unless the operator $U$ is very restricted.

Considering that only the second qubit should be modified, the qubit $|u\rangle$ is the same one. The second has the unknown value $|v_{t+1}\rangle = x|0\rangle + y|1\rangle$.

The tensor product of them is: $|uv_{t+1}\rangle = ax|00\rangle + ay|01\rangle + bx|10\rangle + by|11\rangle$.

Equating $A_U|u\rangle|v\rangle$ with $|uv_{t+1}\rangle$, under the condition that $|u\rangle$ and $|v\rangle$ are not in the basis states, we have that:

$$c = x$$
$$d = y$$
$$c * \cos(\alpha) - d * \sin(\alpha) = x \quad (15)$$
$$c * \sin(\alpha) + d * \cos(\alpha) = y.$$ 

As we can see, the terms of the function of the first qubit disappear of the equations and there are conditions only over the second qubit. One of the conditions is that the qubit before and after the operation of $U$ should be equal. This is only possible if $\alpha = 0$, in other words, $U = I$ (Identity operator) and that $c = d = x = y$, where $|u\rangle = |v\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ are equals and have equal amplitudes in their states.

To be possible to extract the second qubit, it is necessary that the U-Controlled operator be the Identity and the qubits involved be equal. In that...
way, it is not possible to extract the target qubit after any operator $U$, as is the case of qRAM Node and other models.

With that restrictions, how can we recovery a unique value of $z$ of the state of the output qubit during the dynamics if we can not recovery the output qubit $|o\rangle$? The possibility to generate entangled states after some operation is also a problem, once we can not isolate the output after a tangled operation.

![Fig. 7: Filled Julia Set of the edm-ps-qRAM-1 applied to the qRAM](a) $c=3.0j$  
(b) $c=0.5 + 1.0j$

We propose the edm-ps-qRAM model that solves the above mentioned problems. Only some states are to be considered - the elitist states - to the dynamics. In this way, we can follow the dynamics and extract the $z$ complex value loaded in all state. The model has two variance to apply its operators. The first one, edm-ps-qRAM-1, consider the use of a non linear operator $X$ of summation of amplitude and after the quadratic non linear operator $S$ is applied. The second edm-ps-qRAM-2 flips the order of this application: first the quadratic operator is applied and after the summation of amplitude operator is used.

In the step of the sum of amplitudes, the non linear operator $X$ include a imaginary value $c$ in the amplitudes of $|0\rangle$ and $|1\rangle$, as described below:

$$|\psi\rangle = N(z|0\rangle + |1\rangle) \xrightarrow{X} |\psi_x\rangle = N((z + c)|0\rangle + (1 + c)|1\rangle)$$  \hspace{1cm} (16)
Fig. 9: Orbit Diagram of the (a) edm-ps-qRAM-1 and (b) edm-ps-qRAM-2 applied to the qRAM with $-2j < c < 0.25j$.

The second step includes the non linear operator of mixed states [7] described before. For one qubit, the process of this dynamics is:

$$|\psi_x\rangle = N ((z + c)|0\rangle + (1 + c)|1\rangle) \xrightarrow{\Delta z} |\psi_{xs}\rangle = N((z + c)^2|0\rangle + (1 + c)^2|1\rangle)$$ \hspace{1cm} (17)

As the operator qRAM requires $2^n + n + 1$ qubits in the total, for an input qubit of length $n$, for our simplification, we consider a restriction in which states that we will use in that model. The $z$ value is in all state of input. For a qRAM of $n=1$, only the pure states below are considered:

$$|\psi\rangle = N(|0100\rangle + |0101\rangle + z|0110\rangle + |0111\rangle + |1010\rangle + |1011\rangle + z|1110\rangle + |1111\rangle)$$ \hspace{1cm} (18)

These states are chosen because they are the ones that are affected when the qRAM is applied, in other words, they are all activated to apply its operation, due to nature of the CNOT operator that features the qRAM. The other amplitudes are reset to zero to simplify our process to capture $z$ after the dynamic iterations. This simplification of the problem allows us to model the qRAM as a 4 pairs of CNOT gate in the pure state representation.

Repeating the operation explained in the earlier section of the isolating of $z$ value, in an analytical way, we see edm-ps-qRAM-1 Equation as:

$$F_c(z) = \frac{(1 + c)^2}{(z + c)^2}$$ \hspace{1cm} (19)

The second one model edm-ps-qRAM-2 flips the order of the nonlinear operators application, generating this analytical expression:
\[ F_c(z) = \frac{1 + c}{z^2 + c} \]  

(20)

7.1 Analysis

The behaviour of the system is viewed by the Filled Julia Set in the Figure (7) for the model 1 and Figure (8) for the model 2. We can see symmetry and sensibility to initial conditions, the parameter \( c \), in its real and complexity values.

The orbit diagram are drawn and visualized for the \( c \) imaginary values in the Figures (9b) and (9a) for the first and second model respectively. As we can see, the orbit diagrams show easily the chaotic behaviour for a set of \( c \) values and showing the sensibility to initial conditions.

8 Conclusion

The presented model showed that the dynamics of the qRAM weightless quantum neuron node is sensible to initial conditions, exhibiting chaos. The proof that we can not check the second qubit after a CNOT-Operator restricts the use of all states, because we could not recover easily the \( z \) value of the output qubit. The visualization of the experiments helped understanding the behaviour of the dynamics and became an interesting tool to explore the quantum nodes iterations.

Further studies of the dynamics of more robust system with networks of neurons are being pursued. Networks of qRAM are more appropriate for applications in the field of artificial intelligence. The applications of nonlinear and chaos methods seem to be a promising one particularly by the results that nonlinear quantum mechanics can solve NP-complete problems in polynomial many steps [19,20].

There are many follow-up to this research. Other ways to iterate the nodes is being undertaken. As further future work we see the use of other tools from the Dynamical Systems Theory to analyse the models as well a more profound study of the numerical analysis in terms of sensitivity to initial conditions. We can also extend the models of the dynamics to others quantum weightless and weighted neural models [14].

References


