Inverse Gravimetry Problem

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1 Formulation. Non uniqueness

The potential of a (Radon) measure $\mu$ is

$$u(x; \mu) = \int_{\Omega} \Phi(x, y) d\mu(y) \quad (1)$$

where

$$\Phi(x, y) = \frac{1}{4\pi|x - y|},$$

We will assume that $\mu$ is zero outside $\bar{D}$, $D$ is some bounded open set, $\bar{D} \subset \Omega$, $\Omega$ is a given open set in $\mathbb{R}^3$, and $\Gamma_0 \subset \partial \Omega$.

It is known that $-\Delta u(\cdot, \mu) = \mu$

The inverse problem of gravimetry

Find $\mu$ given

$$G_\alpha = \partial^\alpha u(\cdot, \mu), \alpha \in \mathcal{A} \text{ on } \Gamma_0, \quad (2)$$

where $\mathcal{A}$ is some set of multiindices $\alpha$.

A first crucial question is whether there is enough data to (uniquely) find $f$. Let $u_0$ be a function in the Sobolev space $H^2(\Omega)$ with zero Cauchy data $u_0 = \partial_{\nu} u_0 = 0$ on $\Gamma_0$ and let $f_0 = -\Delta u_0$. Due to linearity, $-\Delta(u + u_0) = f + f_0$. Obviously, $u$ and $u + u_0$ have the same Cauchy data on $\Gamma_0$, so $f$ and $f + f_0$ produce the same data (2), but they are different in $\Omega$. It is clear, that there is a very large (infinite dimensional) manifold of solutions to the inverse problems of gravimetry. To regain uniqueness one has to restrict unknown distributions to a smaller but physically meaningful uniqueness class.

Since $f$ with the data (2) is not unique one can look for $f$ with the smallest ($L^2(\Omega)$-) norm. The subspace of harmonic functions $f_h$ is $L^2$-closed, so for any $f$ there is a unique $f_h$
such that \( f = f_h + f_0 \) where \( f_0 \) is \((L^2)\)-orthogonal to \( f_h \). Since the fundamental solution is a harmonic function of \( y \) when \( x \) is outside \( \Omega \), the term \( f_0 \) produces zero potential outside \( \Omega \). Hence the harmonic orthogonal component of \( f \) has the same exterior data and minimal \( L^2 \)-norm. Applying the Laplacian to the both sides of the equation \(-\Delta u(;f_h) = f_h \) we arrive at the biharmonic equation \( \Delta^2 u(;f_h) = 0 \) in \( \Omega \). When \( \Gamma_0 = \partial \Omega \) we have a well-posed first boundary value problem for the biharmonic equation for \( u(;f_h) \). Solving this problem we find from the previous Poisson equation. However, it is hard to interpret \( f_h \) (geo)physically and knowing \( f_h \) does not help much with finding \( f \).

### 2 Uniqueness results

A (geo)physical intuition suggests looking for a perturbing inclusion \( D \) of constant density, i.e. for \( f = \chi_D \) (characteristic function of an open set \( D \)).

Since (in distributional sense) \(-\Delta u(;\mu) = \mu \) in \( \Omega \), by using the Green’s formula (or the definition of a weak solution) we yield

\[
- \int_{\Omega} u^{*} d\mu = \int_{\partial \Omega} ((\partial_\nu u)^{*} - (\partial_\nu u^*)u)
\]

for any function \( u^{*} \in H^1(\Omega) \) which is harmonic in \( \Omega \). If \( \Gamma_0 = \partial \Omega \), then the right side in (3) is known, we are given all harmonic moments of \( \mu \). In particular, letting \( u^{*} = 1 \) we obtain the total mass of \( \mu \), and by letting \( u^{*} \) to be coordinate (linear) functions we obtain moments of \( \mu \) of first order, and hence the center of gravity of \( \mu \).

Even when one assumes that \( f = \chi_D \), there is a non uniqueness due to possible disconnectedness of complement of \( D \). Indeed, it is well known that if \( D \) is the ball \( B(a,R) \) with center \( a \) of radius \( R \) then its Newtonian potential \( u(x,D) = M \frac{1}{4\pi |x-a|} \), where \( M \) is the total mass of \( D \). So the exterior potentials of all annuli \( B(a,R_2) \setminus B(a,R_1) \) are same when \( R_1^3 - R_2^3 = C \) where \( C \) is a positive constant. Moreover, by using this simple example and some reflections in \( \mathbb{R}^n \) one can find two different domains with connected boundaries and equal exterior Newtonian potentials. Augmenting this construction by the condensation of singularities argument from the theory of functions of complex variables one can construct a continuum of different domains with connected boundaries and same exterior potential. So there is a need in geometrical conditions on \( D \).

A domain \( D \) is called star shaped with respect to a point \( a \) if any ray originated at \( a \) intersects \( D \) over an interval. An open set \( D \) is \( x_1 \) convex if any straight line parallel to the \( x_1 \)-axis intersects \( D \) over an interval.

**Theorem 2.1.** Let \( D_1, D_2 \) be two star-shaped with respect to their centers of gravity or two \( x_1 \) convex domains in \( \mathbb{R}^n \). Let \( u_1, u_2 \) be potentials of \( D = D_1, D_2 \).

If \( u_1 = u_2, \partial_\nu u_1 = \partial_\nu u_2 \) on \( \Gamma_0 \), then \( D_1 = D_2 \)

Returning to the uniqueness proof we assume that the are two \( x_1 \)-convex \( D_1, D_2 \) with the same data. By uniqueness in the Cauchy problem for the Laplace equation \( u_1 = u_2 \) near \( \partial \Omega \). Then from (3)

\[
\int_{D_1} u^* = \int_{D_2} u^*
\]
for any function $u^*$ which is harmonic in $\Omega$. The Novikov’s method of orthogonality is to assume that $D_1$ and $D_2$ are different, and then to select $u^*$ in such way that the left integral is less that the right one. To achieve this goal $u^*$ is replaced by its derivative, one integrates by parts to move integrals to boundaries and makes use of the maximum principles to bound interior integrals.

Let $D = \{ x : d(x_1, x_3) < x_2 < \gamma(x_1, x_3) \}$ be a Lipschitz domain. The unknown $d \in Lip$ describes the shape of the floor/ice, $\Gamma = \{(x_1, \gamma(x_1)), x_1 \in \Gamma'\}$ is a known upper part of $\partial D$, $\Gamma_0 = \{ x : x_3 = H \}$ (airborne/satellite measurements).

At present, we have the following uniqueness result.

**Theorem 2.2.** Let $\gamma \in Lip$ and $\Gamma'$ be known. Then the data (2) for $\mu = gd\Gamma + \chi_D$ uniquely determine $g(\text{thickness of snow})$ on $\Gamma$ and $D$.

The exterior gravity field of a polygon (polyhedron) $D$ develops singularities at corner points of $D$. Indeed, $\partial_2 \partial_k u(x; \chi_D)$ where $D$ is a polyhedron with corner at $x_0$ behaves as $-Clog|x - x_0|$, [1], section 4.1. Since these singularities are uniquely identified by the Cauchy data, one has obvious uniqueness results under mild geometrical assumptions on $D$. Moreover use of singularities provides with constructive identification tools, based on range type algorithms in the harmonic continuation, using for example the operator of the single layer potential.

For proofs and further results on inverse problem of gravimetry we refer to work of V. Ivanov, Isakov, Prilepko [1], [5].

### 3 Stability

The inverse problem of potential theory is a severely (exponentially) ill-conditioned problem of the mathematical physics. The character of stability, conditional stability estimates, and regularization methods of numerical solutions of such problems are studied starting from pioneering work of Fritz John and Tikhonov in 1950-60s.

To understand degree of ill-conditioning one can consider harmonic continuation from the circle $\Gamma_0 = \{ x : |x| = R \}$ onto the inner circle $\Gamma = \{ x : |x| = \rho \}$. By using polar coordinates $(r, \phi)$, any harmonic function decaying at infinity can be (in a stable way) approximated by $u(r, \phi; M) = \sum_{m=1}^M u_m r^{-m} e^{im\phi}$. Let us define the linear operator of the continuation as $A(\partial_t(\partial R)) = \partial_t u(\rho)$. Using the formula for $u(\rho; M)$, it is easy to see that the condition number of the corresponding matrix is $(\frac{R}{\rho})^M$ which is growing exponentially with respect to $M$. If $\frac{R}{\rho} = 10$, then use of computers is only possible when $M < 16$, and typical practical measurements errors of 0.01 allow meaningful computational results when $M < 3$.

The following logarithmic stability estimate holds and can be shown to be best possible.

**Theorem 3.1.** Let $D_1, D_2$ be two domains given in polar coordinates $(r, \sigma)$ by the equations $\partial_2 D_j = \{ r = d_j(\sigma) \}$ where $|d_j|_2(S^2) \leq M_2, \frac{1}{M_2} < d_j, j = 1, 2$. Let $\varepsilon = ||u_1 - u_2||_{1}(\Gamma_0) + ||\partial_\sigma(u_1 - u_2)||_{0}(\Gamma_0)$.

Then there is a constant $C$ depending only on $M_2, \Gamma_0$ such that $|d_1 - d_2| \leq C(-log\varepsilon)^{-\frac{1}{2}}$.

A proof in [1] is using some ideas from the proof of Theorem 1.1 and stability estimates for harmonic continuation.
Moreover, while it is not possible to obtain (even local) existence results, a special local existence theorem is available [1], chapter 5. In more detail, if one assumes that \( u_0 \) is a potential of some \( C^3 \)-domain \( D_0 \), that the Cauchy data for a function \( u \) are close to the Cauchy data of \( u_0 \) and that, moreover, \( u \) admits harmonic continuation across \( \partial D_0 \), as well as suitable behavior at infinity, then \( u \) is a potential of a domain \( D \) which is close to \( D_0 \).

Let \( D_{\Gamma} \) be a domain containing \( D \). The gravitational field \( u \) in \( \Omega \setminus D_{\Gamma} \) can be uniquely represented by a single layer potential. Hence the continuation problem from \( \Gamma_0 \) onto \( \Omega \setminus D_{\Gamma} \) reduces to the linear integral equation

\[
\int_{\Gamma} g(y) \partial_x^2 K(x, y) d\Gamma(y) = G_{\alpha}(x), \quad x \in \Gamma_0, \ \alpha \in \mathcal{A}.
\] (4)

For computational purposes we replaced the integral by the finite sum arriving at the linear system \( Ag = G \) for unknown \( g_j \)

\[
\sum_{j=1}^{J} g_j \partial_x^2 K(x(k), y(j)) = G_{\alpha}(x(k)), \quad x(k) \in \Gamma_0, k = 1, \ldots, K
\] (5)

where points \( y(1), \ldots, y(J) \) approximate surface \( \Gamma \) from inside.

When \( \Gamma_0 = \{x_2 = 0\} \) or \( \Gamma_0 = \{x : r = R\} \) one can explicitly continue \( u \) by using the Fourier transformation or spherical harmonics. There are no analytic results when \( \Gamma_0 \) is a (relative small) part of \( \partial \Omega \), so we studied the problem numerically. Comparing results of the continuation from these data one can give recommendation for usefulness of devices measuring higher order derivatives of the gravity potential.

We studied several typical situations when \( \Omega \) is the square \((-2, 2) \times (-2, 2)\), \( \Gamma_0 = (-1.5, 1.5) \times \{0\}\) (green), \( \Gamma \) (blue) is the unit circle, and \( D \) (red) is the rectangle \((-0.25, 0.25) \times (-0.15, 0.15)\)

**Example 2.1 (Gradient)**

<table>
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<th>Number of Msmts (K)</th>
<th>1E-03 noise</th>
<th>1E-06 noise</th>
<th>0E+00 noise</th>
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<td>( k_{\text{min(err)}} )</td>
<td>min(err)</td>
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<td>5.2870E-03</td>
<td>9</td>
<td>3.3336E-04</td>
</tr>
</tbody>
</table>

Table 1: Minimum relative error between calculated and actual field at varying noise levels.
Figure 1: Singular values of the single-layer matrix $A$. 
4 Numerical methods

The idea of the level set (Osher, Set Jian) numerical method of finding \( D = \{ x : d(x) < 0 \} \) is to include \( d \) into a one-parametric family of functions \( d(x, t) \) and to derive for \( d(t) \) an evolution partial differential equation by requirement the derivative of residual with respect to \( t \) is negative. As a result we reformulated our inverse problem as solving the equation \( A(d) = G \) where \( A(d) = \nabla u(\chi_D) \) on \( \Gamma_0 \). Following a general scheme of deriving evolution equation for \( d(t) \) by requirement of decreasing regularized residual

\[
\| A(d(t)) - G \|_2^2(\Gamma_0) + \| \epsilon \nabla d(t) \|_1(\Omega),
\]

with respect to \( t \) we arrive at the following Hamilton-Jacobi type equation

\[
\partial_t d(t) = \epsilon|\nabla d(t)| (\nabla \cdot \frac{\nabla d}{|\nabla d|}) - \int_{\Gamma_0} U(x) \nabla_x K(x, y) d\Gamma(x) \delta(d(y, t)), \tag{6}
\]

where \( H, \delta \) are Heaviside and Dirac delta functions,

\[
\epsilon = \frac{\int_{\Omega} \int_{\Gamma_0} U(x) \nabla_x K(x, y) d\Gamma(x) \delta(d(y, t))(\nabla \cdot \frac{\nabla d}{|\nabla d|})(y, t) dy}{\int_{\Omega} (\nabla \cdot \frac{\nabla d}{|\nabla d|})^2(y, t) dy},
\]

\[
U(x) = \int_{\Omega} \nabla_x K(x, z) H(d(z, t)) dz - G(x).
\]

After discretization with respect to \( t \) we obtain an iterative algorithm.

We report on the reconstruction of three-dimensional inclusions from the data (2) with complete gradient of potential on \( \Gamma_0 = \{ 0 < x_1 < 1, 0 < x_3 < 1, x_2 = 1 \} \). The direct application of the level set method did not produce good results: lower parts of \( D \) were not well reconstructed. So we first used (linear) continuation algorithm to continue the gradient \( G \) of the potential onto the complete boundary of the reference sphere centered at \((0.5, 0.5, 0.5)\) of radius 0.375 and then apply the level set algorithm based on the (6). The data were generated numerically. The data for the inverse problem are given with the relative random error of 0.05. We used 33 grid points in each coordinate. Exact inclusion is red, in blue we show results of iterations of the level set method.
Example 4.1 ( $D$ is two spheres)

Figure 2: Evolution of two spheres.
Example 4.2 ( $D$ is two cubes)

Figure 3: Evolution of two cubes.
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References


